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On S-duality for Non-Simply-Laced Gauge Groups

Philip C. Argyres[1], **Anton Kapustin**[2], **and Nathan Seiberg**[3]

[1] *Physics Department, University of Cincinnati, Cincinnati OH 45221-0011*

[2] *California Institute of Technology, Pasadena, CA 91125*

[3] *School of Natural Sciences, Institute for Advanced Study, Princeton NJ 08540*

Abstract: We point out that for $\mathcal{N} = 4$ gauge theories with exceptional gauge groups G_2 and F_4 the S-duality transformation acts on the moduli space by a nontrivial involution. We note that the duality groups of these theories are the Hecke groups with elliptic elements of order six and four, respectively. These groups extend the $\Gamma_0(3)$ and $\Gamma_0(2)$ subgroups of $SL(2, \mathbb{Z})$ by elements with a non-trivial action on the moduli space. We show that under a certain embedding of these gauge theories into string theory, the Hecke duality groups are represented by T-duality transformations.

Introduction

Strong-weak coupling duality, or S-duality, of $\mathcal{N}=4$ super Yang-Mills (SYM) theory with gauge Lie algebra $\mathbf{g}$ is the conjectured [1, 2] equivalence of this theory to a similar theory with a magnetic-dual Lie algebra $\mathbf{g}^\vee$ and inverse gauge coupling. (We recall the definition of $\mathbf{g}^\vee$ below; mathematicians refer to $\mathbf{g}^\vee$ as the Langlands-dual of $\mathbf{g}$ and denote it ${}^L\mathbf{g}$.) More precisely, let us define the complexified coupling $\tau=(\theta/2\pi)+i(4\pi/g^2)$, where g is the gauge coupling and θ is the theta-angle. For a simply-laced $\mathbf{g}$, we have $\mathbf{g}^\vee=\mathbf{g}$, and strong-weak coupling duality maps τ to $-1/\tau$. There is also a much more obvious symmetry $\tau\to\tau+1$ which corresponds to shifting the theta-angle by 2π. These two transformations together generate the group $SL(2,\mathbb{Z})$ which acts on the coupling τ by fractional linear transformations: $\tau\mapsto(a\tau+b)/(c\tau+d)$, where $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$.[1] The S-duality conjecture is supported by evidence from the invariance of the effective action [3, 4, 5] and the BPS spectrum [6, 7, 8, 4, 9, 10, 11, 12, 13, 14, 15, 16] on the moduli space of the SYM theories, as well as by the transformation properties of the toroidally compactified partition function [17, 18, 19] and of the 't Hooft-Wilson operators [20] in the conformal vacuum. In addition, related checks [21, 22] have been performed for topologically twisted versions of $\mathcal{N}=4$ SYM on more general manifolds, which are sensitive to the gauge group, and not just its Lie algebra.

For the non-simply-laced (compact simple) Lie algebras—B_r, C_r, G_2 and F_4—the situation is more complicated.[2] The magnetic duals of these algebras are $B_r^\vee=C_r$, $C_r^\vee=B_r$, $G_2^\vee=G_2'$, and $F_4^\vee=F_4'$, where the primes on G_2 and F_4 indicate a rotation of their root systems [23] described below. The studies [19, 4, 13, 16, 20] of the partition function, BPS masses, and 't Hooft-Wilson operators for general Lie algebras are consistent with the hypothesis that strong-weak coupling duality maps the theories with Lie algebras $\mathbf{g}$ and $\mathbf{g}^\vee$ to each other and acts by $\tilde{S}:\tau\mapsto\tau^\vee=-1/(q\tau)$ on the coupling. Here q is the ratio of the lengths-squared of long and short roots of the Lie algebra $\mathbf{g}$ or $\mathbf{g}^\vee$ (*i.e.*, $q=2$ for B_r, C_r and F_4, and $q=3$ for G_2). When combined with the 2π shift of the theta-angle, the S-duality group acts on the coupling as an extension of $\Gamma_0(q)\subset SL(2,\mathbb{Z})$ by the generator $\tilde{S}$ [13]. In the case $\mathbf{g}=B_r$ or C_r, since $\tilde{S}$ interchanges the two Lie algebras, it is an equivalence between different theories, and only the $\Gamma_0(2)$ subgroup is the self-duality group. For $\mathbf{g}=G_2$ or F_4, however, the algebras are self-dual so $\tilde{S}$ is supposed to identify strongly-coupled $\mathbf{g}$ with weakly-coupled $\mathbf{g}$. Furthermore, an argument using geometric engineering in type II strings [24] supports the conclusion that G_2 and F_4 are self-dual. (An alternative is that there are new non-Lagrangian $\mathcal{N}=4$ theories which are the strong-coupling limits of the G_2 and F_4 theories.)

The purpose of this note is to sharpen the statement about the action of the conjectural S-duality groups for G_2 and F_4. As pointed out in [13], they are subgroups of $SL(2,\mathbb{R})$ not

[1] The center of $SL(2,\mathbb{Z})$ acts on the theory by charge-conjugation and leaves τ invariant. If the Lie algebra does not have complex representations, then the duality group is $PSL(2,\mathbb{Z})$ rather than $SL(2,\mathbb{Z})$. This is the case for simply-laced Lie algebras $\mathbf{su}(2)$, E_7, and E_8.

[2] We use Dynkin notation for the simple Lie algebras: $A_r=\mathbf{su}(r+1)$, $B_r=\mathbf{so}(2r+1)$, $C_r=\mathbf{sp}(2r)$, $D_r=\mathbf{so}(2r)$.

isomorphic to $SL(2,\mathbb{Z})$. These groups are known as Hecke groups. We note that their actions on the electric and magnetic charge lattices must involve a rotation that is not in the Weyl group. An implication of this is that these S-duality groups not only act on the coupling and the electric and magnetic charges, but also on the moduli space. At self-dual values of the coupling (fixed points of the action of the Hecke groups), this means that certain discrete global symmetries are spontaneously broken at generic points on the moduli space. We also show that the unusual duality groups for G_2 and F_4 are realized as T-duality groups in the string-theoretic approach of [24].

We briefly recall some definitions from the theory of Lie algebras; see *e.g.* [25] for an exposition. At a generic point of the moduli space, the gauge group is Higgsed to $U(1)^r \times \mathcal{W}$, where $\mathcal{W}$ is the Weyl group of $\mathbf{g}$. This breaking is specified by picking a Cartan subalgebra $\mathbf{t} \subset \mathbf{g}$. The (unique up to rescaling) Ad-invariant metric $\langle \cdot, \cdot \rangle$ on $\mathbf{g}$ defines an isomorphism between $\mathbf{t}$ and its dual $\mathbf{t}^*$ by $\langle \alpha, \beta \rangle = \alpha^*(\beta)$ for all $\beta \in \mathbf{t}$. The precise normalization of the metric will be fixed below. We use this metric to identify $\mathbf{t}$ and $\mathbf{t}^*$ and henceforth drop the *'s. The roots $\{\alpha\}$ of $\mathbf{g}$—physically, the $U(1)^r$ charges of the massive gauge bosons—span the root lattice Λ_r in $\mathbf{t}$. The coroots $\{\alpha^\vee\}$ are then defined by $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ and span the coroot lattice $\Lambda_r^\vee$. Physically, the coroots are magnetic charges of elementary BPS monopoles in the theory. It follows from the structure of Lie algebras that the roots belong to the dual of the coroot lattice and *vice versa*, that is, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for all roots α, β. The dual of the root lattice is the magnetic weight lattice, $\Lambda_w^\vee$—the lattice of magnetic charges allowed by the Dirac quantization condition. Thus $\Lambda_w^\vee := \Lambda_r^*$ and $\Lambda_r^\vee \subset \Lambda_w^\vee$. Likewise, the electric charge lattice, or weight lattice, is $\Lambda_w := (\Lambda_r^\vee)^*$ and $\Lambda_r \subset \Lambda_w$. Any state or source is then labelled by its electric and magnetic charges $(\epsilon, \mu) \in \Lambda_w \oplus \Lambda_w^\vee$. The Weyl group $\mathcal{W}$ is a finite group generated by reflections R_α through the planes perpendicular to each root α which act on the electric and magnetic charges by $R_\alpha : (\epsilon, \mu) \mapsto (\epsilon - \langle \alpha^\vee, \epsilon \rangle \alpha, \mu - \langle \mu, \alpha \rangle \alpha^\vee)$. Finally, magnetic dual Lie algebras are defined as follows: $\mathbf{g}^\vee$ is the magnetic dual of $\mathbf{g}$ if its roots are the coroots of $\mathbf{g}$. So $(\mathbf{g}^\vee)^\vee = \mathbf{g}$, $\Lambda_r(\mathbf{g}^\vee) = \Lambda_r^\vee(\mathbf{g})$, and $\Lambda_w(\mathbf{g}^\vee) = \Lambda_w^\vee(\mathbf{g})$. The list of magnetic dual Lie algebras is given in [23].

We also recall some facts related to the action of the $\Gamma_0(q)$ subgroups of $SL(2,\mathbb{Z})$ on the couplings and charges of $\mathcal{N} = 4$ SYM theories [19, 20]. $SL(2,\mathbb{Z})$ is generated by the three elements $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $C = -1$, which satisfy the relations $C^2 = 1$, $S^2 = C$, $(ST)^3 = C$, and C is central. $\Gamma_0(q)$ is the subgroup consisting of the matrices whose lower left entry is a multiple of q. It is generated by C, T, and ST^qS. C is charge conjugation, which acts on charges by $(\epsilon, \mu) \mapsto (-\epsilon, -\mu)$ and leaves the coupling constant invariant. Charge conjugation is a trivial operation for F_4 and G_2 because -1 belongs to the Weyl group. If we choose a normalization of the invariant metric on $\mathbf{g}$ so that short coroots have length $\sqrt{2}$, then the coefficient of the θ parameter in the action is 1 for the minimal instanton, so that θ is periodic with period 2π. T corresponds to the shift of θ by 2π, and so acts by $\tau \mapsto \tau + 1$ and $(\epsilon, \mu) \mapsto (\epsilon + \mu, \mu)$. Finally, ST^qS acts as $\tau \mapsto \tau/(1 - q\tau)$ and $(\epsilon, \mu) \mapsto (-\epsilon, q\epsilon - \mu)$.

We now examine G_2 and F_4 more closely. Details of the root systems of these algebras are tabulated in [26], for example.

$\mathbf{G_2}$

Though the Cartan subalgebra of G_2 is 2 dimensional, it is convenient to describe $\mathbf{t}$ as the plane orthogonal to $e_1 + e_2 + e_3$ in a 3-dimensional space with orthonormal basis $\{e_i\}$, $i = 1, 2, 3$. Then the six short coroots of length $\sqrt{2}$ are $\pm(e_i - e_j)$ for $i \neq j$, and the six long ones of length $\sqrt{6}$ are $\pm(2e_i - e_j - e_k)$ for $i \neq j \neq k$. It follows that the long roots are the same as the short coroots, and the short roots are 1/3 of the long coroots. Therefore, a transformation which takes the roots to the coroots is $R^\vee : e_i \mapsto e_j - e_k$ for $(i, j, k) = (1, 2, 3)$ and cyclic permutations. Upon restriction to the plane orthogonal to $e_1 + e_2 + e_3$, it becomes a rotation by $\pi/2$ accompanied by a rescaling by a factor $\sqrt{3}$.

The Weyl group of G_2 is the dihedral group $\mathcal{D}_6 \simeq \mathcal{S}_3 \ltimes \mathbb{Z}_2$, where the $\mathcal{S}_3$ acts by permutations of the e_i, and the $\mathbb{Z}_2$ by $e_i \mapsto \pm e_i$. Elements of the Weyl group include rotations by $\pi/3$ and reflections, but not $R^\vee/\sqrt{3}$. Note, however, that $(R^\vee)^2/3$, a rotation by π, is an element of the Weyl group.

The moduli space is parametrized by the vacuum expectation values (VEVs) of six real scalars taking values in the Cartan subalgebra, which we write as $\phi = \phi_1 e_1 + \phi_2 e_2 + \phi_3 e_3$ with $\phi_1 + \phi_2 + \phi_3 = 0$.[3] This space should be modded out by the Weyl group. The adjoint Casimirs are a basis of Weyl invariant polynomials in the ϕ_i's. They are clearly symmetric polynomials in the ϕ_i^2. A basis is $s_2 := \sum_i \phi_i^2$ and $s_6 := \prod_i \phi_i^2$, where the subscript on the s_n denotes the scaling dimension. (The dimension four invariant $s_4 := \sum_{i<j} \phi_i^2 \phi_j^2$ is not independent since the $\sum \phi_i = 0$ constraint implies $4s_4 = s_2^2$.) Note that s_2 is determined up to a multiplicative factor by its scaling dimension, while s_6 can be redefined by a multiplicative factor as well as by the addition of a term proportional to s_2^3.

The conjectural $\tilde{S}$ transformation maps the coupling and charges as $\tau \mapsto -1/(3\tau)$ and $(\epsilon, \mu) \mapsto (-R^\vee \mu/3, R^\vee \epsilon)$. Since $R^\vee/\sqrt{3}$ is not an element of the Weyl group, it will have a non-trivial action on the moduli space. Indeed, the BPS mass formula

$$M = \frac{|\phi \cdot (\epsilon + \mu\tau)|}{\sqrt{\mathrm{Im}\tau}}$$

is invariant if in addition to transforming ϵ, μ, and τ as above one maps

$$\tilde{S} : \phi \to \frac{1}{\sqrt{3}} R^\vee \phi.$$

Convenient coordinates on the moduli space are:

$$U_2 = s_2, \quad U_6 := s_6 - (1/54) s_2^3.$$

Then

$$\tilde{S} : (U_2, U_6) \mapsto (U_2, -U_6), \tag{1}$$

[3]The scalars transform as a vector of the $SO(6)_R$ symmetry. For simplicity we consider only invariants made from a single component of this vector.

These coordinates U_2, U_6 which transform homogeneously under $\tilde{S}$ are unique up to overall multiplicative factors.

It is simple to see that the ST^3S generator of $\Gamma_0(3)$ is realized, up to an overall rotation by the element $(R^\vee)^2/3$ of the Weyl group, by $\tilde{S}T\tilde{S}$. (Note that the other natural assignment for the action of $\tilde{S}$ on the charges, namely $(\epsilon, \mu) \mapsto (-(R^\vee)^{-1}\mu, R^\vee \epsilon)$, fails to close on $\Gamma_0(3)$.) This group, generated by C, T, and $\tilde{S}$, is a type of Fuchsian group known as a Hecke group [27]. Its generators satisfy the relations $C^2 = 1$, $\tilde{S}^2 = C$ and $(\tilde{S}T)^6 = C$ with C central.[4] A fundamental domain in the τ-plane is the region $|\tau| \geq 1/\sqrt{3}$ and $|\mathrm{Re}\,\tau| \leq 1/2$ with boundaries identified so that there is a $\mathbb{Z}_2$ orbifold point at $\tau = i/\sqrt{3}$ and a $\mathbb{Z}_6$ orbifold point at $\tau = (i \pm \sqrt{3})/(2\sqrt{3})$.

The G_2 $\mathcal{N} = 4$ SYM theory thus has an enhanced $\mathbb{Z}_2$ and $\mathbb{Z}_6$ global symmetries at these special values of τ. However, at a generic point on the moduli space of vacua these symmetries are spontaneously broken as $\mathbb{Z}_2 \to 1$ and $\mathbb{Z}_6 \to \mathbb{Z}_3$, respectively, by virtue of the action (1).

More generally, if all six Higgs fields are turned on, the moduli space is $\mathbf{t}^{\otimes 6}$ modulo the diagonal action of the Weyl group. The transformation $\tilde{S}$ acts as rotation by $\pi/2$ on all six copies of $\mathbf{t}$. Once one identifies points on the moduli space related by the Weyl group, the $\tilde{S}$ transformation becomes an involution on the moduli space.

$\mathbf{F_4}$

Take $\mathbf{t}$ to be a four-dimensional space with an orthonormal basis $\{e_i\}$, $i = 1, 2, 3, 4$. The 24 short coroots of F_4 are $\pm e_i \pm e_j$ (length $\sqrt{2}$) and the 24 long coroots are $\pm 2e_i$ and $\pm e_1 \pm e_2 \pm e_3 \pm e_4$ (length 2). The long roots are the short coroots, while the short roots are 1/2 the long coroots. A transformation which takes the roots to the coroots is $R^\vee : e_i \mapsto (R^\vee)_i^j e_j$ with

$$R^\vee = \begin{pmatrix} 1 & 1 & & \\ -1 & 1 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{pmatrix}, \tag{2}$$

a rotation by $\pi/4$ in two orthogonal planes together with a rescaling by a factor $\sqrt{2}$.

The Weyl group of F_4 is the group $\mathcal{W} = \mathcal{S}_3 \ltimes (\mathcal{S}_4 \ltimes \mathbb{Z}_2^3)$ where the $\mathcal{S}_4$ in the second factor acts as permutations on the e_i, $(\mathbb{Z}_2)^3$ acts as $e_i \mapsto \pm e_i$ with $\prod_i (\pm)_i = 1$, and the $\mathcal{S}_3$ factor is generated by

$$R_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad R_2 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{3}$$

Note that $R^\vee/\sqrt{2} \notin \mathcal{W}$, but $(R^\vee)^2/2 \in \mathcal{W}$.

[4] Actually, since C is a trivial operation (it belongs to the Weyl group), only the quotient of the Hecke subgroup by its center acts faithfully on the G_2 theory.

An adjoint scalar VEV on the moduli space can be parametrized by $\phi = \sum_i \phi_i e_i$. The Weyl invariant polynomials in the ϕ_i's are clearly symmetric polynomials in the ϕ_i^2 because of the action of the $\mathcal{S}_4 \ltimes \mathbb{Z}_2^3$ factor of $\mathcal{W}$ together with the $\mathbb{Z}_2$ generated by R_2. A basis of these polynomials is $s_2 := \sum_i \phi_i^2$, $s_4 := \sum_{i<j} \phi_i^2 \phi_j^2$, $s_6 := \sum_{i<j<k} \phi_i^2 \phi_j^2 \phi_k^2$, and $s_8 := \prod_i \phi_i^2$. Combinations of these have to be further symmetrized with respect to the $\mathcal{S}_3$ factor generated by R_1 and R_2, giving the four independent $\mathcal{W}$-invariant polynomials

$$\begin{aligned} U_2 &= s_2, \qquad U_6 = 48s_6 - 8s_4 s_2 + s_2^3, \qquad U_8 = 48s_8 - 6s_6 s_2 + 4s_4^2 - s_4 s_2^2, \\ U_{12} &= 1152 s_8 s_4 - 360 s_8 s_2^2 - 216 s_6^2 + 72 s_6 s_4 s_2 - 12 s_6 s_2^3 - 32 s_4^3 + 12 s_4^2 s_2^2 - s_4 s_2^4, \end{aligned} \tag{4}$$

whose dimensions are one plus the exponents of F_4. Note that these U_n are determined up to multiplicative factors and addition of appropriately homogeneous polynomials in the U_m with $m < n$. Such additions could be used to simplify the above formulas for the U_n, but the particular forms shown are chosen to have homogeneous $R^\vee/\sqrt{2}$ transformation properties.

The conjectural $\tilde{S}$ transformation maps coupling and charges as $\tau \mapsto -1/(2\tau)$ and $(\epsilon, \mu) \mapsto (-R^\vee \mu/2, R^\vee \epsilon)$. A straightforward calculation then shows that it acts on the moduli space as

$$\tilde{S} : (U_2, U_6, U_8, U_{12}) \mapsto (U_2, -U_6, U_8, -U_{12}). \tag{5}$$

Unlike the G_2 case, this homogeneous transformation law does not completely determine the Casimirs up to overall rescalings, for U_8 may still be shifted by a multiple of U_2^4, and U_{12} by a multiple of $U_2^3 U_6$.

The ST^2S generator of $\Gamma_0(2)$ is realized by $\tilde{S}T\tilde{S}$. C, T, and $\tilde{S}$ also generate a Hecke group, defined by the relations $C^2 = 1$, $\tilde{S}^2 = C$, and $(\tilde{S}T)^4 = C$, with C central.[5] A fundamental domain in the τ-plane is the region $|\tau| \geq 1/\sqrt{2}$ and $|\mathrm{Re}\,\tau| \leq 1/2$ with boundaries identified so that there is a $\mathbb{Z}_2$ orbifold point at $\tau = i/\sqrt{2}$ and a $\mathbb{Z}_4$ orbifold point at $\tau = (i \pm 1)/2$. Thus the F_4 $\mathcal{N} = 4$ SYM theory has enhanced $\mathbb{Z}_2$ and $\mathbb{Z}_4$ global symmetries at these values of the couplings, which are spontaneously broken to 1 and $\mathbb{Z}_2$, respectively, on the moduli space.

Stringy realization of the duality groups

In [24] it was shown how to embed $\mathcal{N} = 4$ SYM theory with an arbitrary compact simple gauge Lie algebra into string theory so that S-duality follows from a nontrivial symmetry of the worldsheet conformal field theory (essentially, T-duality). This construction provides an alternative way to derive the duality group for G_2 and F_4 theories.

To construct the G_2 theory, one starts with a six-dimensional Little String Theory [28] obtained by taking a decoupling limit of Type IIB string theory on a D_4 ALE singularity. Upon compactification on a circle of radius R_1 this theory becomes equivalent to a five-dimensional $\mathcal{N} = 2$ theory with gauge group $SO(8)$. The coupling constant $1/g_5^2$ of this five-dimensional theory is proportional to R_1. To obtain an $\mathcal{N} = 4$ theory in four dimensions

[5] Again, C acts trivially, so it is the $\mathbb{Z}_2$ quotient of the Hecke group which acts faithfully on the F_4 theory.

with gauge group G_2, one compactifies on a twisted circle of radius R_2 [24]. This means that one considers an orbifold of the five-dimensional theory by a symmetry which acts by the triality automorphism on the five-dimensional fields combined with a translation of x^5 by $2\pi R_2$. Since the triality-invariant part of the Lie algebra of $SO(8)$ is the Lie algebra of G_2, this results in a four-dimensional $\mathcal{N}=4$ theory with gauge group G_2. The coupling $1/g_4^2$ of this theory is proportional to $R_1 R_2$.

To fix the proportionality constant, we note that an instanton in the 4d gauge theory is represented by a Euclidean fundamental string worldsheet wrapping both circles. The action of such a worldsheet instanton is

$$2\pi \frac{R_1 R_2}{\alpha'}.$$

On the other hand, the action of an instanton in gauge theory is $-2\pi i\tau$. Hence we must identify

$$\tau = i\frac{R_1 R_2}{\alpha'} \tag{6}$$

Since τ is purely imaginary, the theta-angle vanishes. To get a nonzero theta-angle, one has to turn on the B-field flux, resulting in

$$\tau = \frac{iR_1 R_2 + B}{\alpha'}.$$

It is shown in [24] that for $B=0$ T-duality along both circles gives the same theory but with

$$R_1' = \frac{\alpha'}{R_1}, \quad R_2' = \frac{\alpha'}{3R_2}.$$

Then we have

$$\tau' = i\frac{\alpha'}{3R_1 R_2} = -\frac{1}{3\tau}.$$

We also have a symmetry $\tau \to \tau + 1$ which corresponds to shifting the B-field flux by α'. These transformations generate a Hecke subgroup of $SL(2,\mathbb{R})$, in agreement with the field-theoretic approach.

The situation for F_4 is similar. One starts with a Little String Theory obtained by considering the decoupling limit of Type IIB string theory on a E_6 ALE singularity and compactifies it on a circle of radius R_1. Then one orbifolds the resulting five-dimensional theory by a transformation which acts by an outer automorphism of E_6 of order 2 and translates x^5 by $2\pi R_2$. This gives a four-dimensional $\mathcal{N}=4$ gauge theory with gauge group F_4 [24]. Its coupling is given by (6). It is shown in [24] that T-duality maps

$$R_1 \mapsto R_1' = \frac{\alpha'}{R_1}, \quad R_2 \mapsto R_2' = \frac{\alpha'}{2R_2}.$$

Hence $\tau' = -1/2\tau$, again in agreement with field theory.

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